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HIGHER-ORDER AVERAGING SCHEMES IN THE THEORY OF NON-LINEAR OSCILLATIONS†

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Non-linear oscillations, described by the standard Bogolyubov system, are investigated by the averaging method. The situation, often encountered in applied problems, when the system of the first approximation does not enable one to judge its essential evolution and qualitative behaviour (the averaged value of the right-hand side of the system is identically equal to zero), is considered. An averaging scheme for describing the evolution over substantially larger time intervals in negative powers of a small parameter (quadratic, cubic, etc.) is proposed and justified. Examples are given which illustrate the effectiveness of the proposed higher-order averaging schemes. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a multidimensional non-linear oscillatory system in the standard Bogolyubov form [1–4]

$$\dot{x} = \varepsilon X(t, x), \quad t \geqslant 0, \quad x(0) = x^0, \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0 \ll 1 \tag{1.1}$$

Here x is is an n-vector, $x \in D \subset R^n$, D is a connected set (usually a closed region [1–3] or an open region [4]), and X is a fairly smooth function of the real variable x, the smoothness properties of which will be refined below. Continuity and 2π-periodicity is assumed with respect to the argument t (where t is the time or a rotating phase).

To investigate the Cauchy problem (1.1), standard constructions of the averaging method are carried out [1–4]. As usual, an averaged system of the first approximation is written

$$\begin{gathered} \dot{\xi} = \varepsilon X_0(\xi), \quad \xi(0) = x^0, \quad X_0(\varepsilon) \equiv \langle X(t, \xi) \rangle \\ \xi = \xi_0(\varepsilon t, x^0) \in D, \quad 0 \leqslant t \leqslant L/\varepsilon, \quad L = \text{const} \end{gathered} \tag{1.2}$$

The angle brackets in (1.2) and henceforth denote averaging over a period of 2π with respect to the explicitly occurring argument t. Then, depending on the nature of the solution of the first approximation ξ (1.2) and the local properties of system (1.2) in the neighbourhood of this solution, the results of the main Bogolyubov theorems regarding the neighbourhood of the solutions (the ε-neighbourhood in the case of periodicity with respect to t) are used in an asymptotically large time interval $0 \leqslant t \leqslant L/\varepsilon$ (the first theorem, and the averaging method [1–6]) or in an unlimited interval (the second theorem and the method of local integral manifolds [1–3]).

If $X_0(\xi) \equiv 0$, the solution $\xi = \xi_0$ (1.2), generally speaking, gives a fairly complete representation of the evolution of the osculating variables x and of the oscillatory process as a whole. Subsequent approximations reduce solely to a small $O(\varepsilon)$ refinement of the solution in the interval $0 \leqslant t \leqslant L/\varepsilon$, which is of no particular importance for investigating the qualitative behaviour of a system both in its theoretical and practical aspects.

We will consider the situation when it is essential to take higher approximations into account. In applied problems we often have the identity $X_0(\xi) \equiv 0$; then $\xi_0 \equiv x^0$ and in the interval $t \sim 1/\varepsilon$ the variable x performs small vibrations with an amplitude $O(\varepsilon)$ about $x = x^0$, and in this case evolution with a velocity $O(\varepsilon^2)$ is possible. We propose another approach, involving the construction of higher-order averaging schemes with respect to ε and a considerable increase in the range of change of the argument t. To do this we use a standard transformation of the variable $x \to \xi$, used as a basis for the averaging method or the change of variables [1–6]

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$$x = \xi + \varepsilon u(t, \xi), \quad u \equiv \int_0^t X(s, \xi) ds, \quad X_0(\xi) \equiv 0 \tag{1.3}$$

By differentiating expressions (1.3), taking into account the identity for u, we obtain relations of the form

$$\dot{\xi} = \varepsilon^2 \Xi(t, \xi, \varepsilon), \quad \xi(0) = x^0 \quad (\partial u / \partial t \equiv X(t, \xi)) \tag{1.4}$$

$$\Xi \equiv (I + \varepsilon u'_\xi)^{-1} \varepsilon^{-1} [X(t, \xi + \varepsilon u) - X(t, \xi)], \quad \xi \in D$$

The right-hand side of system (1.4), i.e. the function Ξ, for fairly small values of $\varepsilon > 0$ will be continuous with respect to ξ, $\xi \in D$, and 2π-periodic with respect to t; the following approximate representation holds for Ξ

$$\Xi = \Xi_{(0)}(t, \xi) + \varepsilon \Xi_1(t, \xi) + \varepsilon^2 \ldots \equiv \Xi_{(0)} + \varepsilon \Delta \Xi$$

$$\Xi_{(0)} = X'_\xi u, \quad \Xi_1 = \frac{1}{2}(X''_{\xi^2} u, u) - u'_\xi X'_\xi u \tag{1.5}$$

For the Cauchy problem (1.4), (1.5) a range of the argument $0 \leqslant t \leqslant L/\varepsilon^2$ is considered, in which the variable ξ may change considerable – by an amount of the order of unity. Then the initial variable x changes by the same amount. It is required to construct a higher-order scheme in powers of ε and to substantiate the estimate of closeness. Note that the main purpose of this asymptotic approach, like the classical Krylov–Bogolyubov method [1], is to avoid singular terms in the approximate construction of the solution in the range of variation of the argument $0 \leqslant t \leqslant L/\varepsilon^2$ considered. The need to develop such an approach in its computational aspect is also important, since numerical integration of Cauchy problems (1.1), (1.4) when $t \sim 1/\varepsilon^2$ is even more problematic than when $t \sim 1/\varepsilon$.

2. AVERAGING OVER AN EXTENDED INTERVAL

We will apply to the Cauchy problem (1.4), (1.5) the standard scheme change of variables $\xi \to \eta$ of the type (1.3)

$$\xi = \eta + \varepsilon^2 v(t, \eta, \varepsilon) = \eta + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 \ldots$$

$$v \equiv \int_0^t (\Xi(s, \eta, \varepsilon) - \langle \Xi \rangle) ds, \quad \eta \in D \tag{2.1}$$

$$\dot{\eta} = \varepsilon^2 \Xi_0(\eta) + \varepsilon^2 \Xi_{10}(\eta) + \varepsilon^4 \mathrm{H}(t, \eta, \varepsilon), \quad \eta(0) = x^0$$

$$\Xi_0 = \langle \Xi_{(0)}(t, \eta) \rangle, \quad \Xi_{10} = \langle \Xi_1(t, \eta) \rangle, \ldots, \quad 0 \leqslant t \leqslant L/\varepsilon^2$$

Here H is a known 2π-periodic function of t, fairly smooth and uniformly bounded with respect to η, $\eta \in D$, for sufficiently small values of $\varepsilon > 0$; its definition is similar to (1.4) for Ξ. Suppose $\Xi_0 \equiv 0$; then it is natural to introduce an interval of the argument $0 \leqslant t \leqslant L/\varepsilon^2$ in which the variable η, and together with it also the variables ξ and x, vary by a considerable amount of the order of unity. We will neglect terms $O(\varepsilon^3)$ and higher on the right-hand side of system (2.1). We obtain an autonomous system of the "first approximation", the solution of which is assumed to be constructed (analytically or numerically)

$$\dot{\eta}_0 = \varepsilon^2 \Xi_0(\eta_0), \quad \eta_0(0) = x^0, \quad \eta_0 = \eta_0(\varepsilon^2 t, x^0)$$

$$0 \leqslant \varepsilon^2 t \leqslant L, \quad \Xi_0 = \langle X'_h u \rangle \tag{2.2}$$

The function Ξ_0 in (2.1) and (2.2) is defined by (1.5); the quantities $X(t, \xi)$, $u(t, \xi)$ are taken with $\xi = \eta = \eta_0$. An estimate of the uniform closeness of the solutions of problems (2.1) and (2.2) is made using integral inequalities (Gronwall's lemma [1–4]). In fact, the standard procedure for constructing estimates of the averaging method leads to the uniform limit

$$\max_t|\eta-\eta_0|\leqslant C_\eta\varepsilon,\quad 0\leqslant t\leqslant L/\varepsilon^2;\quad C_\eta,\ L-\text{const}$$
$$C_\eta=ML\exp(\lambda L),\quad M=\max_{t,\eta}|\Xi_{10}+\varepsilon H|,\quad \eta\in D,\quad 0\leqslant\varepsilon\leqslant\varepsilon_0 \tag{2.3}$$

Here λ is the Lipschitz constant with respect to η of the function $\Xi_0(\eta)$.

It is assumed [1–3], that $\eta_0\in D$ together with a certain small neighbourhood in the range of variation of t considered, according to (2.3). It follows directly from (2.3), taking (2.1) and (1.3) into account, that the solution η_0 of (2.2) is ε-close to the solutions ξ, x of problems (1.4) and (1.1) respectively

$$\max_t|\xi-\eta_0|\leqslant C_\xi\varepsilon,\quad \max_t|x-\eta_0|\leqslant C_x\varepsilon,\quad 0\leqslant t\leqslant L/\varepsilon^2$$
$$(C_{\xi,x}-\text{const}) \tag{2.4}$$

Hence, when X_0 ω 0, $\Xi_0\equiv 0$ the solution of the first approximation η_0 (2.2) defines, in the interval $t\sim 1/\varepsilon^2$, the evolution of system (1.1) with a small error $O(\varepsilon)$ (see (2.4)). The procedure for refining the solution η of problem (2.1) in powers of ε can be realized similarly, as is the case in the averaging method [1–6]. The required degree of accuracy is limited by the smoothness of the function Ξ, i.e. the initial right-hand side X of (1.1). In particular, if the function $\Xi_{(0)}$ satisfies the Lipschitz condition with respect to ξ, $\xi\in D$, with a uniformly bounded constant λ, while the function $\Delta\Xi$ is uniformly bounded in the region $0\leqslant\varepsilon\leqslant\varepsilon_0$, $\eta\in D$, $t\geqslant 0$, then, for the solution of the first approximation η_0 (2.2) we have estimates of the ε-closeness of (2.3) and (2.4). The function $\Xi_{(0)}$ will be such, if the initial function X is continuously differentiable with respect to x, $x\in D$, while the derivatives satisfy the Lipschitz condition. Naturally, a scheme of higher order in powers of ε than the standard averaging scheme requires greater smoothness. It is essentially connected with the construction of the second approximation (see Section 3).

If the function $\Xi_{(1)}=\Xi_{(0)}+\varepsilon\Xi_1$ (1.5) satisfies the Lipschitz condition with respect to ε, i.e. the second derivatives of the function X satisfy this condition with respect x, $x\in D$, then, by the change of variables (2.1), we can write the system of equations of the "second approximation", which leads to an error $O(\varepsilon^2)$ in the extended range of variation of the argument considered. The refined Cauchy problem has the form

$$\dot\eta_{(1)}=\varepsilon^2\Xi(\eta_{(1)})+\varepsilon^3\Xi_{10}(\eta_{(1)}),\quad \eta_{(1)}(0)=x^0$$
$$\eta_{(1)}=\eta_0(\varepsilon^2t,x^0)+\varepsilon\eta_1(\varepsilon^2t,x^0)+\varepsilon^2\ldots,\quad 0\leqslant\varepsilon^2t\leqslant L \tag{2.5}$$

The function η_1 in (2.5) is found using numerical or analytical methods based on the generating solution η_0 (2.2) and the variational system [2]. Here it is more convenient to introduce the slow argument $\tau=\varepsilon^2t$, $0\leqslant\tau\leqslant L$. Without loss of accuracy in powers of ε we can substitute $\eta_0(\tau,x^0)$ to the function $\Xi_{(0)}$. Using the solution $\eta_{(1)}(\tau,x^0,\varepsilon)$ (2.5) obtained, we have the limits

$$\max_t|\eta-\eta_{(1)}|\leqslant C_\eta\varepsilon^2,\quad \max_t|\xi-\eta_{(1)}|\leqslant C_\xi\varepsilon^2$$
$$\max_t|x-[\eta_{(1)}+\varepsilon u(t,\eta_0)]|\leqslant C_x\varepsilon^2,\quad 0\leqslant t\leqslant L/\varepsilon^2 \tag{2.6}$$

The construction of more accurate solutions $\eta_{(k)}(t,x^0,\varepsilon)$, $k\geqslant 3$, which differ from η by $O(\varepsilon^k)$ for $0\leqslant t\leqslant L/\varepsilon^2$, requires greater smoothness of the functions X and extremely complex expansions, taking the expression for $\mathrm{H}(t,\eta,\varepsilon)$ (2.1) into account (see below). It is usually of little interest: corrections do not make any appreciable contribution to the evolution of the system in the interval $0\leqslant t\leqslant L/\varepsilon^2$. We have the following assertion.

Theorem 1. When the condition $X_0(x)\equiv 0$ (1.2) is satisfied as well as the conditions for smoothness with respect to x and periodicity with respect to t of the function $X(t,x)$ formulated above, the solution $x(t,x^0,\varepsilon)$ of system (1.1) is described in the interval $0\leqslant t\leqslant L/\varepsilon^2$ by the functions $\eta_0(\varepsilon^2t,x^0)$ (2.2) with an error $O(\varepsilon)$, according to (2.4), and the function $\eta_{(1)}+\varepsilon u(t,\eta_0)$ (2.5) with an error $O(\varepsilon^2)$, according to (2.6). When $\Xi_0\not\equiv 0$, in general, the variable x varies by an amount δx of the order of unity with respect to the small parameter ε: $|\delta x|=|x-x^0|\sim O(1)$, $t\sim 1/\varepsilon^2$.

The proof of Theorem 1 was in fact carried out in the preceding constructions and has a constructive form. Note that the function X and the quantity x^0 may depend continuously on the small parameter

ε. This assumes that expansions in powers of ε are carried out [1–3] (see the examples in Sections 4.2 and 4.3). In particular, when $X = X(t, x, \varepsilon), x(0) = x^0(\varepsilon)$, we must put in (1.2)–(1.4)

$$X_0(\xi) \equiv \langle X(t, \xi, 0)\rangle, \quad \xi(0, \varepsilon) = x^0(0), \quad u \equiv \int_0^t X(s, \xi, 0)ds \tag{2.7}$$

The last expressions in (1.4) and (1.5) and others require that the functions $X'_\varepsilon, X''_{\varepsilon^2}, \ldots$ be taken into account and refinement of the quantities $\xi(0, \varepsilon)$, i.e. $x^{0\prime}(0)$, $x^{0\prime\prime}(0)$, ... must be taken into account. Of course, if the dependence of the functions X and x^0 on ε is not smooth (for example, continuous), these expansions are impossible and the dependence is taken into account completely.

3. HIGHER-ORDER PROCEDURES OF THE AVERAGING METHOD

Together with the first-order identity $X_0(\xi) \equiv 0$, a second-order identity $\Xi_0(\eta) \equiv 0$ may also occur, i.e.

$$\Xi_0(\eta) = \langle X'_x(t, \eta)\int_0^t X(s, \eta)ds\rangle \equiv 0 \tag{3.1}$$

The variable η, in the first approximation, then does not vary in the interval $t \sim 1/\varepsilon^2$, and by (2.2) we obtain $\eta_0 = x^0$. As a result $|\eta - x^0| = O(\varepsilon)$ (see (2.3)). The variables x and η also vary by an amount $O(\varepsilon)$ when $t \sim 1/\varepsilon^2$. By analogy with the above, we establish that the evolution of the slow variable η is determined by the quantity $O(\varepsilon^3)$ (see (2.1) and (1.5))

$$\begin{gathered}\dot\eta = \varepsilon^3\Xi_{10}(\eta) + \varepsilon^4 \mathrm{H}(t, \eta, \varepsilon), \quad \eta(0) = x^0 \\ \Xi_{10}(\eta) = \tfrac{1}{2}\langle (X''_{x^2}(t, \eta)u(t, \eta), u(t, \eta))\rangle - \langle u'_\xi(t, \eta)X'_x(t, \eta)u(t, \eta)\rangle\end{gathered} \tag{3.2}$$

The procedure of the method of separation of motions is applicable to system (3.2) if the right-hand side is sufficiently smooth with respect to η, and this leads to a "first approximation" system and to the limits

$$\begin{gathered}\dot\zeta_0 = \varepsilon^3\Xi_{10}(\zeta_0), \quad \zeta_0(0) = x^0, \quad \zeta_0 = \zeta_0(\varepsilon^3 t, x^0) \\ |\eta - \zeta_0| \leqslant C_\eta\varepsilon, \quad |\xi - \zeta_0| \leqslant C_\xi\varepsilon, \quad |x - \zeta_0| \leqslant C_x\varepsilon, \quad 0 \leqslant t \leqslant L/\varepsilon^3\end{gathered} \tag{3.3}$$

Hence, when $X_0 = \Xi_0 \equiv 0$, a considerable change in the variable x occurs in the interval $t \sim 1/\varepsilon^3$ if $\Xi_{10} \not\equiv 0$. The first approximation (with an error $O(\varepsilon)$) is determined by the function ξ_0 (3.3). A more accurate calculation requires, as usual, making use of the standard scheme of the averaging method based on explicit expressions for the functions $\Xi_{10}(\eta)$ and $\mathrm{H}(t, \eta, \varepsilon)$. We can similarly construct a system of equations of the evolution of the slow variables of any power of ε. Note that the proposed procedure of successive changes of variables was used in [5]

Higher-order schemes can be realized in another way starting from the standard procedure of the averaging method, namely, by changing the variables for the initial system (1.1), described previously [1–3]

$$\begin{gathered}x = \xi + \varepsilon U(t, \xi, \varepsilon) = \xi + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots + \varepsilon^k U_k + \varepsilon^{k+1} \ldots \\ U_i = U_i(t, \xi), \quad U_i(0, \xi) = 0 \\ \dot\xi = \varepsilon\Theta(\xi, \varepsilon) = \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^2\Theta_3 + \ldots + \varepsilon^k\Theta_k + \varepsilon^{k+1} \ldots \\ \xi(0) = x^0, \quad \Theta_i = \Theta_i(\xi)\end{gathered} \tag{3.4}$$

The coefficients U_i of the asymptotic expansion (3.4) and the corresponding expressions for Θ_i are calculated in the required number in terms of the derivatives of the function X at the point $x = \xi$, by quadratures and algebraic operations. If we confine ourselves to a k-th order expansion, the variable ξ will be described by the equation

$$\dot{\xi} = \varepsilon\Theta_{(k)}(\xi,\varepsilon) + \varepsilon^{k+1}\Theta^*_{k+1}(t,\xi,\varepsilon), \quad \xi(0) = x^0$$
$$\Theta_{(k)} = \Theta_1 + \varepsilon\Theta_2 + \ldots + \varepsilon^{k-1}\Theta_k \tag{3.5}$$

Suppose $\Theta_1(\xi) \not\equiv 0$; then the system of the first approximation, which defines the evolution of the vector x, is described by relations (3.4) and (3.5) when $k = 1$

$$x = \xi + \varepsilon U_1(t,\xi), \quad \dot{\xi} = \varepsilon\Theta_1(\xi) + \varepsilon^2\Theta^*_2(t,\xi,\varepsilon), \quad \xi(0) = x^0$$
$$U_1 = \int_0^t [X(s,\xi) - \Theta_1(\xi)]ds, \quad \Theta_1 = \langle X(t,\xi)\rangle \not\equiv 0 \tag{3.6}$$

Discarding $\varepsilon^2\Theta^*_2$ and solving the Cauchy problem, we obtain a solution of the first approximation $\xi_1(\varepsilon t, x)$. Expansions of the type (3.4), (3.5) lead to a refinement of the initial solution x by an amount $O(\varepsilon)$ when $t \sim 1/\varepsilon$.

Suppose now that $\Theta_1(\xi) \equiv 0$, $\Theta_2(\xi) \not\equiv 0$. Then, it is natural to take expressions (3.4) and (3.5) as the system of the first approximation when $k = 2$ (see Section 2)

$$x = \xi + \varepsilon U_1(t,\xi) + \varepsilon^2 U_2(t,\xi), \quad U_1 = \int_0^t X(s,\xi)ds$$
$$\dot{\xi} = \varepsilon^2\Theta_2(\xi) + \varepsilon^2\Theta^*_3(t,\xi,\varepsilon), \quad \xi(0) = x^0 \tag{3.7}$$
$$\Theta_2 = \langle X'_x(t,\xi)U_1(t,\xi)\rangle, \quad U_2 = \int_0^t [X'_x(s,\xi)U_1(s,\xi) - \Theta_2]ds$$

We will take the function ξ_1 as the solution of the first approximation of system (3.7). This function is defined by the relations (see (2.2))

$$\dot{\xi}_1 = \varepsilon^2\Theta_2(\xi_1), \quad \xi_1(0) = x^0, \quad \xi_1 = \xi_1(\varepsilon^2 t, x^0), \quad 0 \leqslant t \leqslant L/\varepsilon^2$$
$$|\xi - \xi_1| \leqslant C_\xi\varepsilon, \quad |x - \xi_1| \leqslant C_x\varepsilon \tag{3.8}$$

If higher powers of ε are taken into account in relations (3.4) and (3.5), this leads to small $O(\varepsilon)$ corrections to the initial solution x (see (3.2) and (3.3)).

The system of the first approximation of an arbitrary k-th power in ε can be written down similarly

$$x = \xi + \varepsilon U_1 + \ldots + \varepsilon^k U_k, \quad \Theta_1 = \Theta_2 = \ldots = \Theta_{k-1} \equiv 0, \quad \Theta_k \not\equiv 0$$
$$\dot{\xi}_1 = \varepsilon^k\Theta_k(\xi_1), \quad \xi_1(0) = x^0, \quad \xi_1 = \xi_1(\varepsilon^k t, x^0) \tag{3.9}$$
$$|\xi - \xi_1| \leqslant C_\xi\varepsilon, \quad |x - \xi_1| \leqslant C_x\varepsilon, \quad 0 \leqslant t \leqslant L/\varepsilon^k$$

Refinement of the initial solution x requires the construction and integration of an averaged system for ξ of higher order than the k-th approximation. As already pointed out, these calculations can be carried out fairly simply using the perturbation method. It is then more convenient to introduce the slow argument $\tau_k = \varepsilon^k t$, $0 \leqslant \tau_k \leqslant L$.

Theorem 2. Suppose, when using the averaging method in the k-th approximation with respect to ε, the identities $\Theta_1 = \Theta_2 = \ldots \Theta_{k-1} \equiv 0$ are satisfied, but $\Theta_k(\xi) \not\equiv 0$, $k \geqslant 1$ can have any value. Then, the qualitative evolution of system (1.1) occurs in the interval $0 \leqslant t \leqslant L/\varepsilon^k$ and is described by the function $\xi_1(\tau_k, x^0)$ (3.9) with an error $O(\varepsilon)$.

The closeness of x and ξ_1 when $0 \leqslant t \leqslant L/\varepsilon^k$ is proved using integral inequalities (Gronwall's lemma [1–4]).

Note that in addition to using the standard procedure of the averaging method [1–6], one can use a Newton-type recurrent accelerated convergence method to construct the averaged system. At the first stage of the iteration procedure we assume, according to transformation (1.3)

$$x = x_{(1)} + \varepsilon U_1(t, x_{(1)}), \quad U_1 = \int_0^t X(s, x_{(1)})ds, \quad \langle X \rangle \equiv 0$$

$$\dot{x}_{(1)} = \varepsilon^2 X_{(1)}(t, x_{(1)}, \varepsilon) \tag{3.10}$$

$$X_{(1)} = \varepsilon^{-1}(I + \varepsilon U'_{1x})^{-1}[X(t, x_{(1)} + \varepsilon U_1) - X(t, x_{(1)})]$$

If the average $\langle X_{(1)} \rangle = O(\varepsilon^2)$ or smaller in powers of ε, the following iteration step is carried out

$$x_{(1)} = x_{(2)} + \varepsilon^2 U_2(t, x_{(2)}, \varepsilon), \quad U_2 = \int_0^t X_{(1)}(s, x_{(2)}, \varepsilon)ds$$

$$\dot{x}_{(2)} = \varepsilon^4 X_{(2)}(t, x_{(2)}, \varepsilon), \quad \varepsilon^4 = \varepsilon^{(2^2)} \tag{3.11}$$

$$X_{(2)} = \varepsilon^{-2}(I + \varepsilon^2 U'_{2x})^{-1}[X_{(1)}(t, x_{(2)} + \varepsilon^2 U_2, \varepsilon) - X_{(1)}(t, x_{(2)}, \varepsilon)]$$

Similarly, if the average $\langle X_{(2)} \rangle = O(\varepsilon^4)$ or smaller in powers of ε, we obtain in a similar way the averaged system of the third approximation $\dot{x}_{(3)} = O(\varepsilon^8)$. At the k-th step we have the system (provided $\langle X_{(k-1)} \rangle = O(\varepsilon^{\theta(k-1)})$ or less)

$$\dot{x}_{(k)} = \varepsilon^{\theta(k)} X_{(k)}(t, x_{(k)}, \varepsilon), \quad \theta(k) = 2^k \tag{3.12}$$

The evolution of the variable x is determined from system (3.12) if the average $\langle X_{(k)} \rangle = O(\varepsilon^K)$, where $0 \leqslant K \leqslant \theta(k) - 1$. The evolution equations of the first approximation are reduced to the form (3.9). A considerable change in the variable x occurs in the range of the argument $0 \leqslant \tau \leqslant L\varepsilon^K$, $\tau = \varepsilon^{\theta(k)}t$.

4. EXAMPLES

We will consider non-linear oscillatory systems, an investigation of the essential evolution of which requires the use of an averaging scheme of the second order in ε (see Sections 2 and 3).

4.1. *A model example.* As an illustration we will take the scalar equation, which also allows of analytical integration,

$$\dot{x} = \varepsilon \sin(t + x), \quad x(0) = x^0, \quad X_0 \equiv 0 \tag{4.1}$$

$$\operatorname{arctg} \frac{\operatorname{tg} z/2 + \varepsilon}{\sqrt{1 - \varepsilon^2}} \Bigg|_{z=x^0}^{z=x+t} = \frac{t}{2}\sqrt{1 - \varepsilon^2}$$

By expanding relation (4.1) between x and t, ε in powers of ε or using the standard averaging procedure [1–6], we obtain the required solution of the Cauchy problem – an expression for x in the second approximation in ε

$$x = \xi_{(1)}(\tau, x^0) + \varepsilon(\cos x^0 - \cos(x^0 + t)) + O(\varepsilon^2) = x^0 + O(\varepsilon) \tag{4.2}$$

$$0 \leqslant t \leqslant L/\varepsilon$$

$$\xi_{(1)} \equiv x^0 - \tau/2, \quad \tau = \varepsilon^2 t \quad (0 \leqslant \tau \leqslant L\varepsilon), \quad \delta x = x - x^0 \sim \varepsilon$$

We apply the averaging scheme of the second order to Cauchy problem (4.1) (see Section 2 and 3); we obtain the required solution in the first and second approximations

$$x = \xi_{(1)}(\tau, x^0) + \varepsilon[\cos\xi^0 - \cos(t + \xi_{(1)})] + O(\varepsilon^2) = \xi_{(1)}(\tau, x^0) + O(\varepsilon), \quad 0 \leqslant t \leqslant L/\varepsilon^2 \tag{4.3}$$

Comparison of expressions (4.3) and (4.2) indicates that the first term of the expansion determines the solution of the second order with an error $O(\varepsilon)$ in the interval $t \sim 1/\varepsilon^2$. After a time $0 \leqslant t \leqslant L/\varepsilon^2$ a considerable evolution δx of the variable x occurs (by an amount $\delta x \approx -L/2$), which is not obvious from (4.1) and (4.2). The fact that a term $O(\varepsilon^2)$ occurs in (4.2) in the interval $t \sim 1/\varepsilon$ by no means guarantees, that it will be the case for the interval $t \sim 1/\varepsilon_2$ (for example, when $O(\varepsilon^2) = O(\varepsilon^2(\varepsilon t)^2)$.

4.2. *Quasi-linear oscillations of a mechanical system.* Consider the forced oscillations of a non-linear oscillator of the form

$$\ddot{q}+Q(q)=P(t)-\Lambda\dot{q},\quad q(0)=q^0,\quad \dot{q}(0)=\dot{q}^0 \tag{4.4}$$

Here Q is a non-linear restoring force in the neighbourhood of the equilibrium position $q = q^*$ ($q^* = 0$), P is a two-frequency periodic force, and Λ is the coefficient of viscous friction. Suppose $Q'(0) = \omega^2 > 0$ and let us assume that the frequencies $\nu_{1,2}$ of the external force are commensurable with ω, i.e. a resonance situation occurs. Then, assuming the quantities q^0, $\dot{q}^0$, Λ and the amplitude $|P|$ to be small, which we will characterize by the parameter ε, we can reduce Eq. (4.4) to the form (after cancelling ε)

$$\begin{aligned}
&\ddot{y}+\omega^2 y=\varepsilon F(t,y,\dot{y},\varepsilon),\quad y(0)=a,\quad \dot{y}(0)=v\\
&F\equiv h\sin\nu_1 t+\alpha y^2+\varepsilon(f\sin(\nu_2 t+\varphi)+\beta y^3-\lambda\dot{y})+\varepsilon^2\gamma y^4+\varepsilon^3\ldots,\\
&q=\varepsilon y,\quad P=\varepsilon^2(h\sin\nu_1 t+\varepsilon f\sin(\nu_2 t+\varphi)),\quad 0<\varepsilon\ll 1\\
&\alpha=-\frac{1}{2}Q''(0),\quad \beta=-\frac{1}{6}Q'''(0),\quad \gamma=-\frac{1}{24}Q^{\mathrm{IV}}(0),\quad \Lambda=\varepsilon^2\lambda>0
\end{aligned} \tag{4.5}$$

We will further assume that $\nu_1 = 2\omega$, $\nu_2 = \omega$, while the remaining parameters are arbitrary quantities of the order of unity. We will introduce the dimensionless time ($\omega t \to t$) and appropriately redenote the parameters. Then, changing to the Van der Pol osculating variable $x = (x_1, x_2)^T$, we obtain a system of equations of the type (1.1) in the standard Bogolyubov form [1–3]

$$\begin{aligned}
&\dot{x}=\varepsilon X(t,x,\varepsilon),\quad X_1=-F\sin t,\quad X_2=F\cos t\\
&x_1(0)=a,\quad x_2(0)=v\\
&y=x_1\cos t+x_2\sin t,\quad \dot{y}=-x_1\sin t+x_2\cos t
\end{aligned} \tag{4.6}$$

Note that the function X depends on ε, i.e., it is necessary to take expression (2.7) into account. It follows from (4.5) and (4.6) that $X_{(0)}(x) = \langle X(t, x, 0)\rangle$, i.e. the situation described in Section 1 arises. The slow variable x may undergo considerable evolution $\delta x \sim O(1)$ in the interval $t \sim 1/\varepsilon^2$. A system of a higher, second-order type (1.4) is constructed by making change of variables (1.3), in which

$$\begin{aligned}
&u_1=-\frac{\alpha}{3}[\xi_1^2(1-c^3)+2\xi_1\xi_2 s^3+\xi_2^2(2-3c+c^3)]-\frac{2}{3}hs^3,\quad c\equiv\cos t\\
&u_2=\frac{\alpha}{3}[\xi_1^2(3s-s^3)+2\xi_1\xi_2(1-c^3)+\xi_2^2 s^3]+\frac{2}{3}h(1-c^3),\quad s\equiv\sin t
\end{aligned} \tag{4.7}$$

In addition to the term $X'_x(t, \xi, 0)u(t, \xi)$ the vector $X'_\varepsilon(t, \xi, 0)$ also occurs in the expression for Ξ, due to the correction $O(\varepsilon)$ in the function F (4.5). Dropping quantities $O(\varepsilon^3)$ in the equation for ξ, we obtain a system of the second order in ε

$$\dot{\xi}=\varepsilon^2(X'_x(t,\xi,0)u(t,\xi)+X'_\varepsilon(t,\xi,0)),\quad \xi(0)=x^0 \tag{4.8}$$

The next stage of investigating system (4.6) involves the averaging of the right-hand sides of Eqs (4.8) over t (see (2.1)). In the first approximation, we obtain for the averaged variable η

$$\begin{aligned}
&\dot{\eta}=\varepsilon^2(\langle(X'_x)\rangle u+\langle(X'_\varepsilon)\rangle),\quad \eta(0)=x^0,\quad 0\leqslant t\leqslant L/\varepsilon^2\\
&\langle(X'_{1x})u\rangle=-\frac{5}{12}\alpha^2\eta^2\eta_2+\frac{1}{6}\alpha h\eta_1\\
&\langle(X'_{1\varepsilon})\rangle=-\frac{1}{2}f\cos\varphi-\frac{3}{8}\beta\eta^2\eta_2-\frac{1}{2}\lambda\eta_1\\
&\langle(X'_{2x})u\rangle=\frac{5}{12}\alpha^2\eta^2\eta_1-\frac{1}{6}\alpha h\eta_2\\
&\langle(X'_{2\varepsilon})\rangle=\frac{1}{2}f\sin\varphi+\frac{3}{8}\beta\eta^2\eta_1-\frac{1}{2}\lambda\eta_2
\end{aligned} \tag{4.9}$$

The expressions in parentheses of the type (X) denote that the function X is taken with $x = \eta$ and $\varepsilon = 0$. System (4.9) possesses obvious structural properties; it can be effectively investigated by qualitative phase-plane methods. In particular, when there is no external force ($h = f = 0$) Eqs (4.9) can be integrated in terms of elementary functions

$$\eta^2 = x^{02}\exp(-\varepsilon\lambda t), \quad \eta^2 = \eta_1^2 + \eta_2^2, \quad x^{02} = a^2 + v^2 \tag{4.10}$$

Expression (4.10) represents a decrease in the total energy of system (4.5) with time, due to linear dissipation. After substituting η^2 from (4.10) into (4.9) we obtain an elementary integrable system

$$\dot{\eta} = \varepsilon^2\left(\frac{5}{12}\alpha^2 + \frac{3}{8}\beta\right)\eta^2 J\eta - \frac{\varepsilon^2}{2}\lambda\eta, \quad J = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \tag{4.11}$$

$$\eta_1 = (a\cos\psi - v\sin\psi)\exp\left(-\frac{1}{2}\varepsilon^2\lambda t\right)$$

$$\eta_2 = (a\sin\psi + v\cos\psi)\exp\left(-\frac{1}{2}\varepsilon^2\lambda t\right)$$

$$\psi = \left(\frac{5}{12}\alpha^2 + \frac{3}{8}\beta\right)\lambda^{-1}(1 - \exp(-\varepsilon^2\lambda t))$$

According to relations (4.11), the motion of a passive system in the time interval $t \sim 1/\varepsilon^2$ reduces to slow rotation of the vector of the initial state with an exponentially decreasing angular velocity and a slow exponential decay. There is a stationary point of the stable node type in the system (in the linear approximation). It is interesting to note that the non-linear perturbations $O(\varepsilon y^2)$ and $O(\varepsilon^2 y^3)$ in (4.5) in the interval $t \sim 1/\varepsilon^2$ lead to actions $O(\varepsilon^2\eta^2\eta_{1,2})$ of the same structure (see (4.9) and (4.11)). They have a conservative character, where the first is independent of the sign of α while the second is determined by the quantity β, which may lead to their mutual compensation. The harmonic force $\varepsilon h \sin 2t$ (4.5), for sufficiently large $|\alpha h|$ leads to unstable saddle-type stationary points (in particular when $f = 0$). When $h = 0$ we obtain an oscillatory system, which is extremely interesting from the mechanical point of view, similar to a Duffing oscillator, which has been investigated in some detail in the literature, including other resonance relations [7–10]. Hence, the approach described confirms that in the interval $1/\varepsilon^2$ interesting evolutionary processes occur in system (4.5), which are not found in standard investigations in the interval $t \sim 1/\varepsilon$.

4.3. *Perturbed parametric oscillations of the second order.* Consider the motion of a plane physical pendulum, whose axis undergoes single-frequency oscillations, taking into account the moment of the viscous forces. Unlike the case of rapid vibrations [1–3] we will assume that the frequency of small oscillations is comparable with the excitation frequency. The amplitude of the oscillations of the suspension point and the moment of the viscous-friction forces will be assumed to be relatively small quantities. We have the equation of motion

$$J\ddot{\varphi} + Mgl\sin\varphi = -Ml(\ddot{x}_0(\nu t)\cos\varphi + \ddot{y}_0(\nu t)\sin\varphi) - \Lambda\dot{\varphi} \tag{4.12}$$

Here φ is the angular deflection of the axis of the pendulum from the vertical, J is the moment of inertia, M is the mass, l is the distance between the suspension point and the centre of mass along the axis of the pendulum, (x_0, y_0) are the coordinates of the suspension point, Λ is the coefficient of the moment of the viscous-friction forces, and ν is the excitation frequency. To fix our ideas we will assume that the point moves along an ellipse in the plane of oscillations of the pendulum

$$x_0 = d\sin(2\nu t + \delta), \quad y_0 = h\cos 2\nu t; \quad d, h, \nu, \delta, \Lambda = \text{const} \tag{4.13}$$

Further, using expression (4.13), we will introduce the dimensionless argument θ and parameters N^2, χ, ε, λ; we obtain, instead of (4.12), the equation

$$\varphi'' + (N^2 - \varepsilon\cos 2\theta)\sin\varphi = \chi\sin(2\theta + \delta)\cos\varphi - \lambda\varphi' \tag{4.14}$$

$$\theta = \nu t, \quad N^2 = Mgl(J\nu^2)^{-1}, \quad \chi = 4MldJ^{-1}$$

$$\varepsilon = 4MlhJ^{-1}, \quad \lambda = \Lambda(J\nu)^{-1}$$

Here the prime denotes a differentiation with respect to the argument θ. We will take the quantity ε as the small parameter; we will also assume χ and λ to be small while the quantities N and δ, generally speaking, are arbitrary.

Consider the case of resonant quasi-linear oscillations of system (4.14), which lead to a second-order averaging scheme. The corresponding assumptions and equation have the form

$$y''+(4-\varepsilon\cos 2\theta)y=\varepsilon^2[\varkappa\sin(2\theta+\delta)-4\gamma y+\frac{2}{3}y^3-\sigma y']+O(\varepsilon^3) \tag{4.15}$$

$$\varphi=\varepsilon y,\quad N=2+\varepsilon^2\gamma,\quad \chi=\varepsilon^3\varkappa,\quad \lambda=\varepsilon^2\sigma;\quad 0<\varepsilon\ll 1$$

Here y, γ, x, σ are quantities of the order of unity.

An essential difference between Eqs (4.15) and (4.5) is parametric excitation, which is of the order of ε. Further constructions are carried out in the same way as in Section 4.2. The change to osculating variables x_1 and x_2 is carried out using formulae of the type (4.6). It was established that the averages of the functions X_1 and X_2 with respect to θ when $\varepsilon = 0$ are identically equal to zero, while the functions u_1 and u_2 and the averaged system of the second order have the form

$$\begin{aligned}
u_1&=-\frac{\xi_1}{12}(1-\cos^3 2\theta)-\frac{\xi_2}{12}\sin^3 2\theta\\
u_2&=\frac{\xi_1}{12}(3\sin 2\theta-\sin^3 2\theta)+\frac{\xi^2}{12}(1-\cos^3 2\theta)\\
\xi_1'&=\varepsilon^2\left(\left(\frac{1}{192}+\gamma\right)\xi_2-\frac{\varkappa}{4}\cos\delta-\frac{1}{8}\xi^2\xi_2-\frac{1}{2}\sigma\xi_1\right),\quad \xi_1(0)=x_1^0\\
\xi_2'&=\varepsilon^2\left(\left(\frac{5}{192}-\gamma\right)\xi_1+\frac{\varkappa}{4}\sin\delta+\frac{1}{8}\xi^2\xi_1-\frac{1}{2}\sigma\xi_2\right),\quad \xi_2(0)=x_2^0
\end{aligned} \tag{4.16}$$

The solution of autonomous system (4.16) $\xi_{1,2}(\varepsilon^2\theta, x_1^0, x_2^0)$ defines the variables $x_{1,2}$ and y, y' with an error $O(\varepsilon)$ in the range $0\leqslant\theta\leqslant L/\varepsilon^2$. It can be investigated fairly completely by phase-plane methods. Note that the terms $\pm\gamma\xi_{1,2}$ and $\mp\xi^2\xi_{1,2}$ have a conservative form, i.e., they occur in the equation for the total energy. The terms $-\sigma\xi_{1,2}/2$ lead to an energy that decreases exponentially to zero, but terms of the same sign (the parametric action) and terms containing the parameters x and δ (external resonance action), will hinder this dissipation.

To be specific, we will consider the situation when there are no horizontal oscillations of the suspension point ($\varkappa = 0$). The system then has a stationary point $\xi_1 = \xi_2 = 0$; we will investigate its stability in the sense of the averaged equations. The characteristic exponents are

$$p_{1,2}=-\frac{\sigma}{2}\pm\mu,\quad \mu=\left[\left(\frac{1}{192}+\gamma\right)\left(\frac{5}{192}-\gamma\right)\right]^{1/2} \tag{4.17}$$

It can be seen that the expression $\mu(\gamma)$ (4.17) vanishes when $\gamma_1 = -1/192$ and $\gamma_2 = 5/192$. Outside the range $\gamma < \gamma_1$, $\gamma > \gamma_2$ the expression for μ is imaginary, and consequently this stationary point is asymptotically stable ($\sigma > 0$, a stable focus), while the ε-closeness between $\xi_{1,2}$ and $x_{1,2}$ will exist for all $\theta \geqslant 0$ [1–3]. Inside the range of the function $\mu \geqslant 0$, a maximum is reached: $\mu^* = 164$ and $\gamma^* = 1/96$. If the dimensionless dissipation factor σ in (4.17) is sufficiently large ($\sigma > 1/32$), then $p_{1,2} < 0$, and we will have a stable mode and the stated conclusions on closeness. If $0 < \sigma < 1/32$, a set of values of γ exists, where the necessary and sufficient conditions for asymptotic stability hold (a node or a focus). There is a range of values $\gamma' < \gamma < \gamma''$ on the γ axis, to which an exponentially unstable point (a saddle) corresponds; the boundary points $\gamma = \gamma', \gamma''$ correspond to the critical case of a single zero root ($p_1 = 0$); the other root is negative ($p_2 < 0$). Hence, the extension of the scheme of the investigations to the interval $\theta \sim 1/\varepsilon^2$ enables us to investigate, as in the example in Section 4.2, extremely interesting features of the evolution of the oscillations of a pendulum with a slowly oscillating suspension point in the ε^2-neighbourhood of the (2:2) resonance regime considered.

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